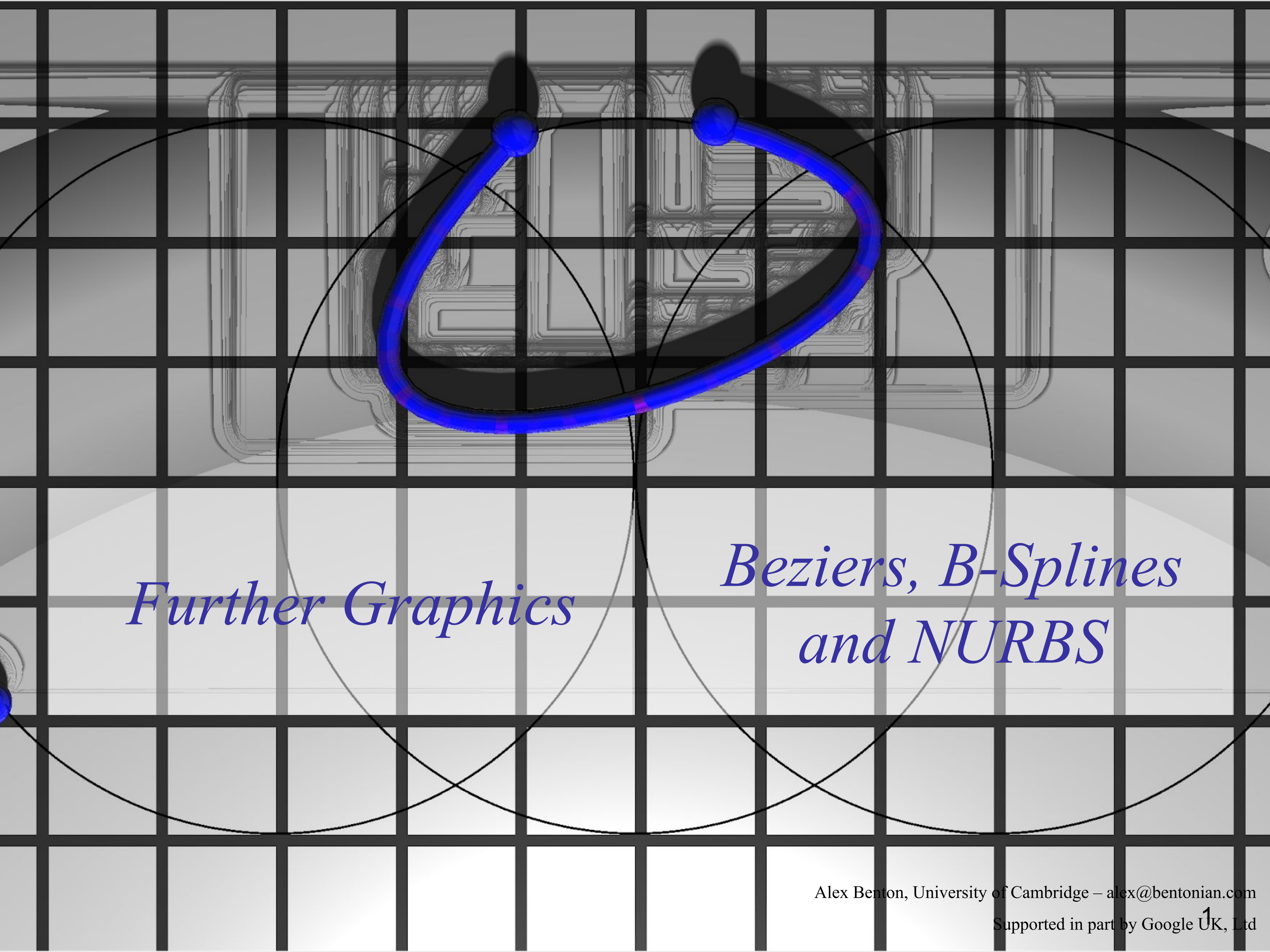
Further Graphics
Beziers, B-Splines and NURBS
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CAD, CAM, and a new motivation: **shiny things**

Expensive products are sleek and smooth.

→ Expensive products are C2 continuous.

Shiny, but reflections are warped

Shiny, and reflections are perfect

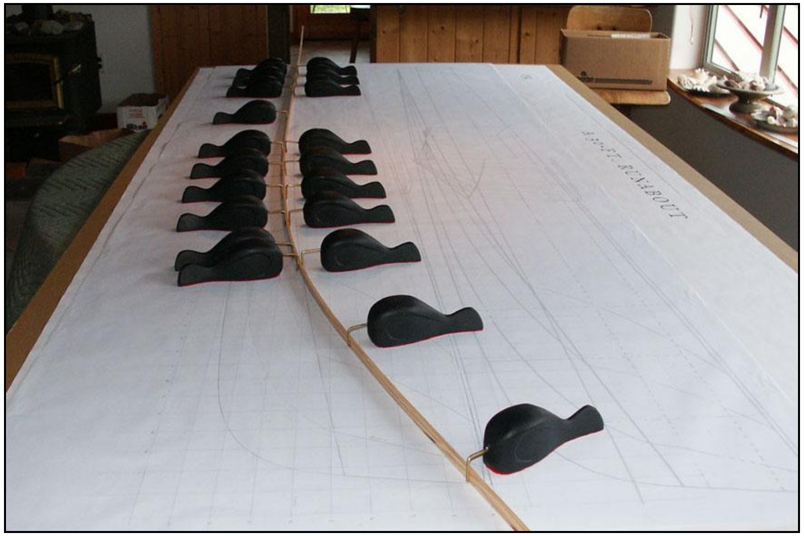
The drive for smooth CAD/CAM

- *Continuity* (smooth curves) can be essential to the perception of *quality*.
- The automotive industry wanted to design cars which were aerodynamic, but also visibly of high quality.
- Bezier (Renault) and de Casteljau (Citroen) invented Bezier curves in the 1960s. de Boor (GM) generalized them to B-splines.

History

The term *spline* comes from the shipbuilding industry: long, thin strips of wood or metal would be bent and held in place by heavy ‘ducks’, lead weights which acted as control points of the curve.

Wooden splines can be described by C_n-continuous Hermite polynomials which interpolate n+1 control points.



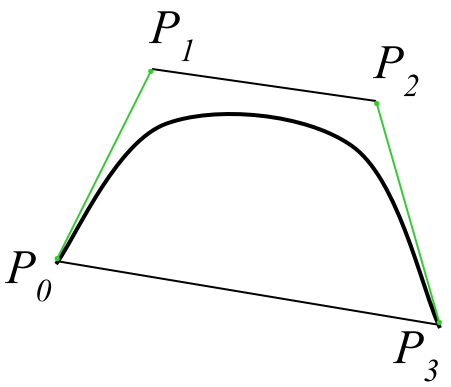
Top: Fig 3, P.7, Bray and Spectre, *Planking and Fastening*, Wooden Boat Pub (1996)

Bottom: http://www.pranos.com/boatsofwood/lofting%20ducks/lofting_ducks.htm

Bezier cubic

- A *Bezier cubic* is a function P(t) defined by four control points:

$$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$



- P_0 and P_3 are the endpoints of the curve
- P_1 and P_2 define the other two corners of the bounding polygon.
- The curve fits entirely within the convex hull of $P_0 \ldots P_3$.

Beziers

Cubics are just one example of Bezier splines:

- Linear: $P(t) = (1-t)P_0 + tP_1$
- Quadratic: $P(t) = (1-t)^2P_0 + 2t(1-t)P_1 + t^2P_2$
- Cubic: $P(t) = (1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + t^3P_3$

...

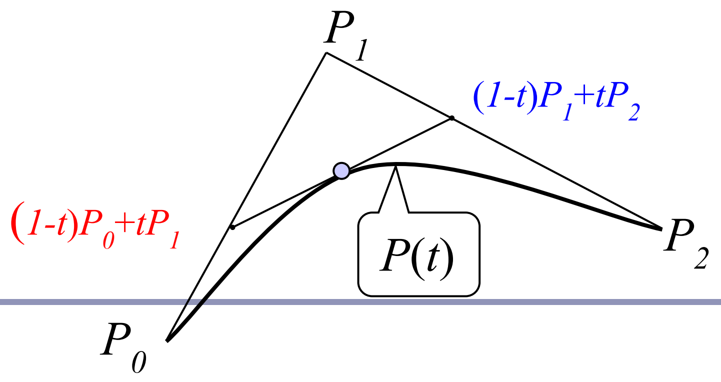
"*n choose i*" = $n! / i!(n-i)!$

General:

$$P(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_i, \; 0 \le t \le 1$$

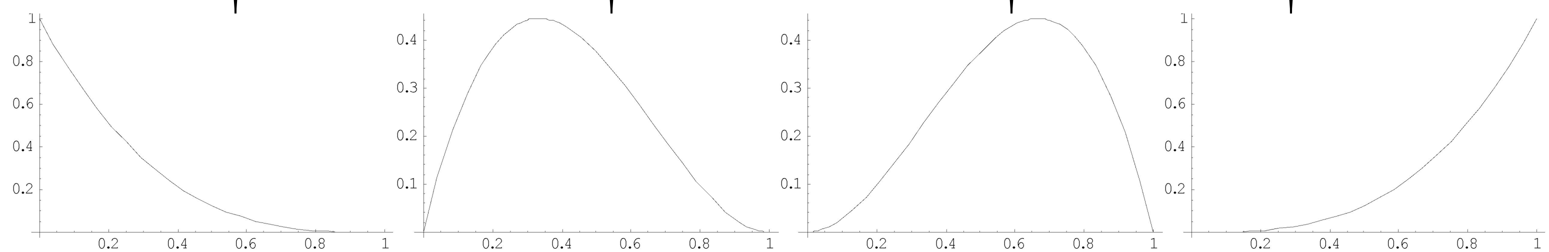
Beziers

- You can describe Beziers as *nested linear interpolations:*
  - The linear Bezier is a linear interpolation between two points:
    $P(t) = (1\text{-}t)\ (P_0) + (t)\ (P_1)$
  - The quadratic Bezier is a linear interpolation between two lines:
    $P(t) = (1\text{-}t)\ ((1\text{-}t)P_0+tP_1) + (t)\ ((1\text{-}t)P_1+tP_2)$
  - The cubic is a linear interpolation between linear interpolations between linear interpolations… etc.
- Another way to see Beziers is as a *weighted average* between the control points.



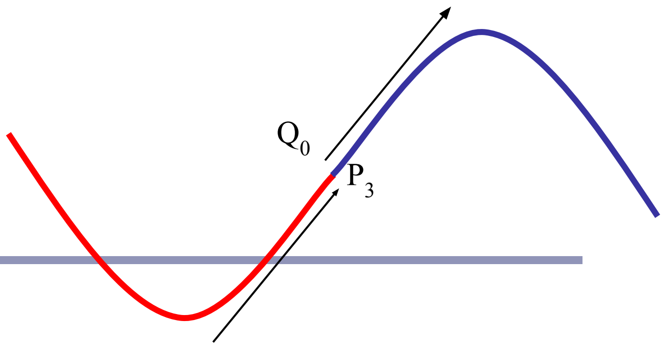
Bernstein polynomials

$$P(t) = \underbrace{(1\text{-}t)^3}P_0 + \underbrace{3t(1\text{-}t)^2}P_1 + \underbrace{3t^2(1\text{-}t)}P_2 + \underbrace{t^3}P_3$$



- The four control functions are the four *Bernstein polynomials* for *n=3*.
  - General form: $b_{v,n}(t) = \binom{n}{v} t^v (1-t)^{n-v}$
  - Bernstein polynomials in $0 \leq t \leq 1$ always sum to 1:
    $$\sum_{v=1}^{n} \binom{n}{v} t^v (1-t)^{n-v} = (t + (1-t))^n = 1$$

Types of curve join

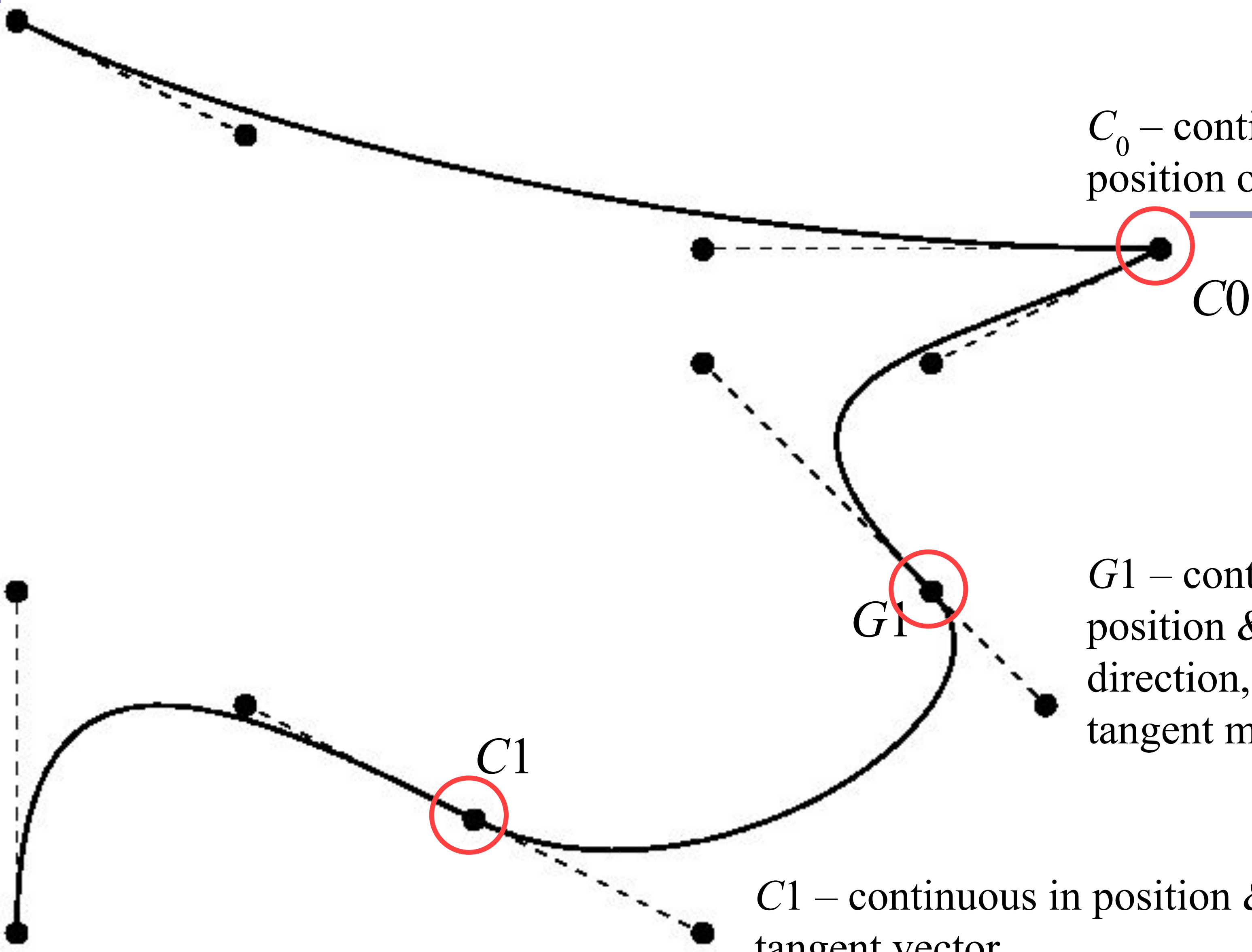


- each curve is smooth within itself
- joins at endpoints can be:
  - C_1 – continuous in both position and tangent vector
    - smooth join in a mathematical sense
  - G_1 – continuous in position, tangent vector in same direction
    - smooth join in a geometric sense
  - C_0 – continuous in position only
    - “corner”
  - discontinuous in position

C_n (mathematical continuity): continuous in all derivatives up to the n^{th} derivative

G_n (geometric continuity): each derivative up to the n^{th} has the same “direction” to its vector on either side of the join

$C_n \Rightarrow G_n$



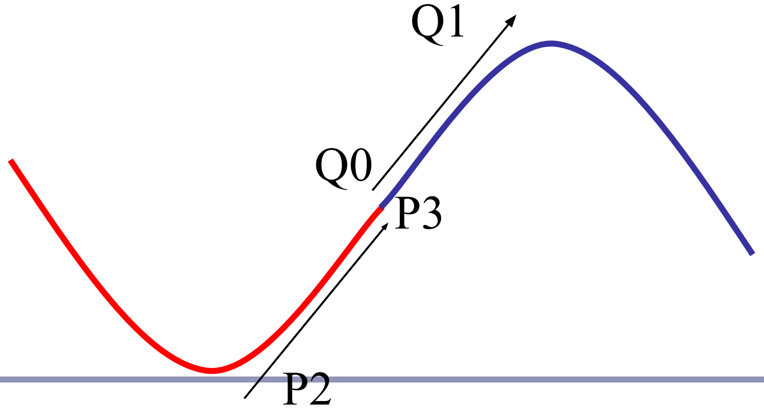
C_0 – continuous in position only

$G1$ – continuous in position & tangent direction, but not tangent magnitude

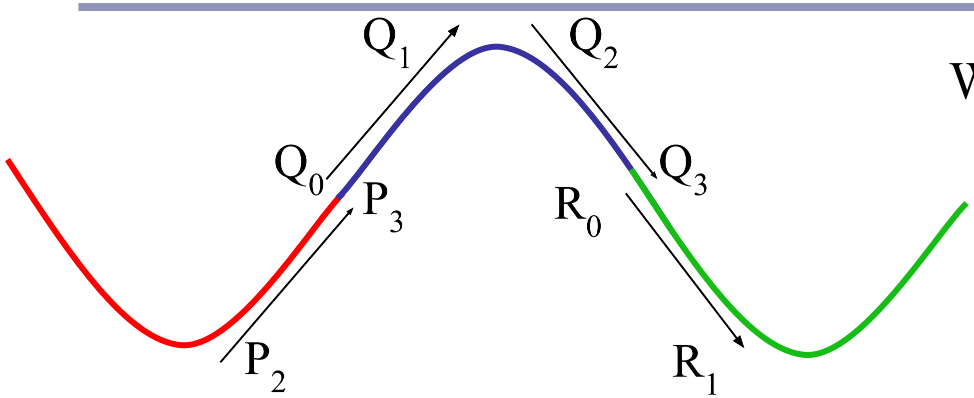
$C1$ – continuous in position & tangent vector

Joining Bezier splines

- To join two Bezier splines with C0 continuity, set $P_3=Q_0$.
- To join two Bezier splines with C1 continuity, require C0 and make the tangent vectors equal: set $P_3=Q_0$ and $P_3-P_2=Q_1-Q_0$.



What if we want to chain Beziers together?



Consider a chain of splines with many control points…

$P = \{P_0, P_1, P_2, P_3\}$
$Q = \{Q_0, Q_1, Q_2, Q_3\}$
$R = \{R_0, R_1, R_2, R_3\}$

…with C1 continuity…

$P3=Q_0$, $P_2-P_3=Q_0-Q_1$
$Q3=R_0$, $Q_2-Q_3=R_0-R_1$

We can parameterize this chain over *t* by saying that instead of going from 0 to 1, *t* moves smoothly through the intervals [0,1,2,3]

The curve $C(t)$ would be:

$$C(t) = P(t) \cdot ((0 \le t < 1)\ ?\ 1 : 0) + Q(t-1) \cdot ((1 \le t < 2)\ ?\ 1 : 0) + R(t-2) \cdot ((2 \le t < 3)\ ?\ 1 : 0)$$

[0,1,2,3] is a type of *knot vector*. 0, 1, 2, and 3 are the *knots*.

B-Splines

B-Splines ("Basis Splines") are a generalization of Beziers. B-splines are built from a <u>series of splines</u>, joined with known continuity.

- A B-spline curve is defined between t_{min} and t_{max}:

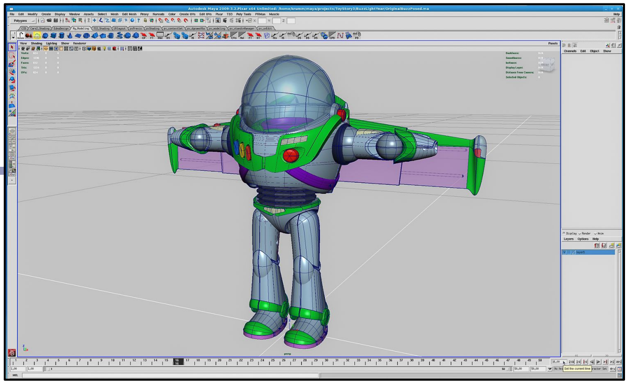
$$P(t) = \sum_{i=1}^{n} N_{i,k}(t) P_i, \;\; t_{min} \le t < t_{max}$$

- $N_{i,k}(t)$ is the *basis function* of control point P_i for parameter k. $N_{i,k}(t)$ is defined recursively:

$$N_{i,1}(t) = \begin{cases} 1, \; t_i \le t < t_{i+1} \\ 0, \; \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

B-Splines



B-splines are defined by:

- $\{P_1 \ldots P_n\}$, te list of n *control points*
- d, the *degree* of the curve
- $k = d+1$, called the *parameter* of the curve
- $[t_1, \ldots, t_{k+n}]$, a *knot vector* of (k+n) parameter values (“knots”)

A B-spline curve will have the following traits:

- $d = k\text{-}1$ is the degree of the curve, so k is the number of control points which influence a single interval
  - Ex: a cubic (d=3) has four control points (k=4)
- There are $k+n$ knots t_i, and $t_i \leq t_{i+1}$ for all t_i
- Each B-spline is $C_{(k\text{-}2)}$ continuous: *continuity* is degree minus one, so a k=3 curve has d=2 and is C1

http://www.mikekrummhoefener.com/toy-story-char-grid/

B-Splines

$t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad \ldots$

$k=1$: $N_{1,1}(t) \quad N_{2,1}(t) \quad N_{3,1}(t) \quad N_{4,1}(t) \quad \ldots$

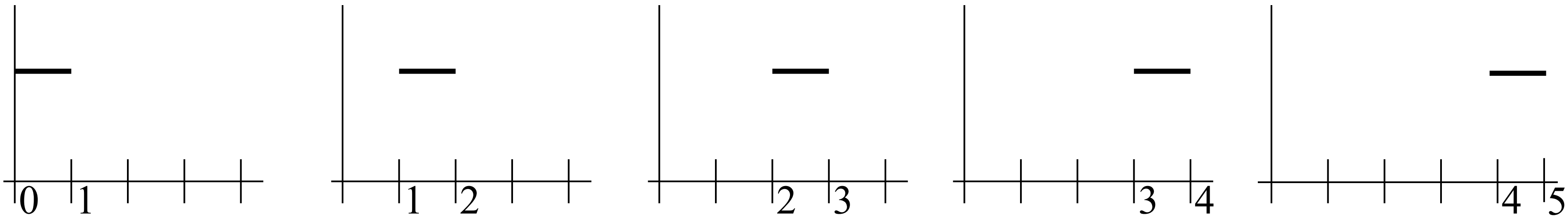
$k=2$: $N_{1,2}(t) \quad N_{2,2}(t) \quad N_{3,2}(t) \quad \ldots$

$k=3$: $N_{1,3}(t) \quad N_{2,3}(t) \quad \ldots$

$k=4$: $N_{1,4}(t) \quad \ldots$

$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

B-Splines



$N_{1,1}(t)$ $N_{2,1}(t)$ $N_{3,1}(t)$ $N_{4,1}(t)$ $N_{5,1}(t)$

$t_1 = 0.0$
$t_2 = 1.0$
$t_3 = 2.0$
$t_4 = 3.0$
$t_5 = 4.0$
$t_6 = 5.0$

$N_{1,1}(t)=1,\ 0 \le t < 1$

$N_{3,1}(t)=1,\ 2 \le t < 3$

$N_{5,1}(t)=1,\ 4 \le t < 5$

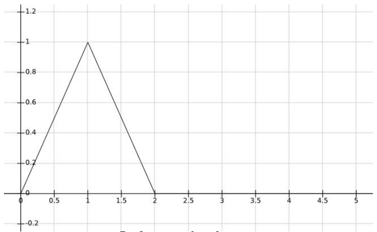
$N_{2,1}(t)=1,\ 1 \le t < 2$

$N_{4,1}(t)=1,\ 3 \le t < 4$

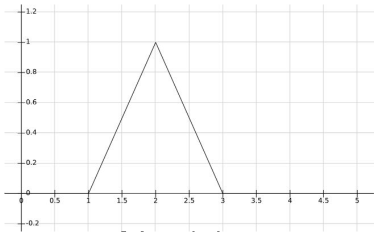
Knot vector = $\{0,1,2,3,4,5\}$, $k = 1 \rightarrow d = 0$ (degree = zero)

B-Splines

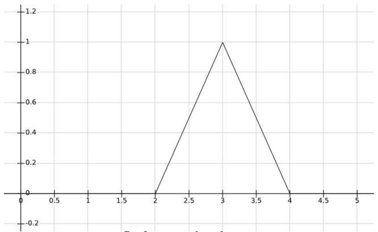
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$



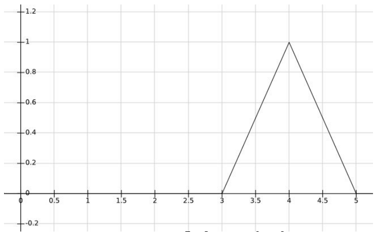
$N_{1,2}(t)$



$N_{2,2}(t)$



$N_{3,2}(t)$



$N_{4,2}(t)$

$$N_{1,2}(t) = \frac{t-0}{1-0} N_{1,1}(t) + \frac{2-t}{2-1} N_{2,1}(t) = \begin{cases} t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \end{cases}$$

$$N_{2,2}(t) = \frac{t-1}{2-1} N_{2,1}(t) + \frac{3-t}{3-2} N_{3,1}(t) = \begin{cases} t-1 & 1 \leq t < 2 \\ 3-t & 2 \leq t < 3 \end{cases}$$

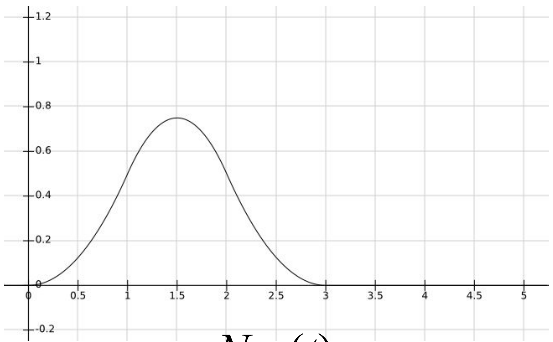
$$N_{3,2}(t) = \frac{t-2}{3-2} N_{3,1}(t) + \frac{4-t}{4-3} N_{4,1}(t) = \begin{cases} t-2 & 2 \leq t < 3 \\ 4-t & 3 \leq t < 4 \end{cases}$$

$$N_{4,2}(t) = \frac{t-3}{4-3} N_{4,1}(t) + \frac{5-t}{5-4} N_{5,1}(t) = \begin{cases} t-3 & 3 \leq t < 4 \\ 5-t & 4 \leq t < 5 \end{cases}$$

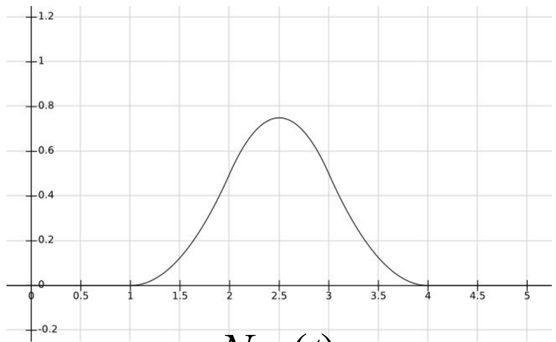
Knot vector = {0,1,2,3,4,5}, $k = 2 \rightarrow d = 1$ (degree = one)

B-Splines

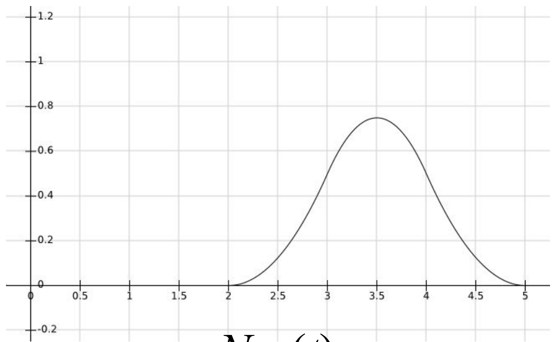
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$



$N_{1,3}(t)$



$N_{2,3}(t)$



$N_{3,3}(t)$

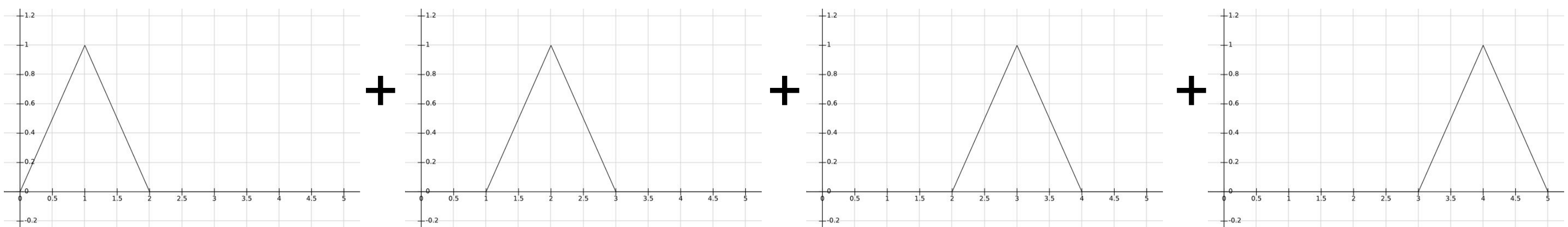
$$N_{1,3}(t) = \frac{t-0}{2-0} N_{1,2}(t) + \frac{3-t}{3-1} N_{2,2}(t) = \begin{cases} t^2/2 & 0 \le t < 1 \\ -t^2 + 3t - 3/2 & 1 \le t < 2 \\ (3-t)^2/2 & 2 \le t < 3 \end{cases}$$

$$N_{2,3}(t) = \frac{t-1}{3-1} N_{2,2}(t) + \frac{4-t}{4-2} N_{3,2}(t) = \begin{cases} (t-1)^2/2 & 1 \le t < 2 \\ -t^2 + 5t - 11/2 & 2 \le t < 3 \\ (4-t)^2/2 & 3 \le t < 4 \end{cases}$$

$$N_{3,3}(t) = \frac{t-2}{4-2} N_{3,2}(t) + \frac{5-t}{5-3} N_{4,2}(t) = \begin{cases} (t-2)^2/2 & 2 \le t < 3 \\ -t^2 + 7t - 23/2 & 3 \le t < 4 \\ (5-t)^2/2 & 4 \le t < 5 \end{cases}$$

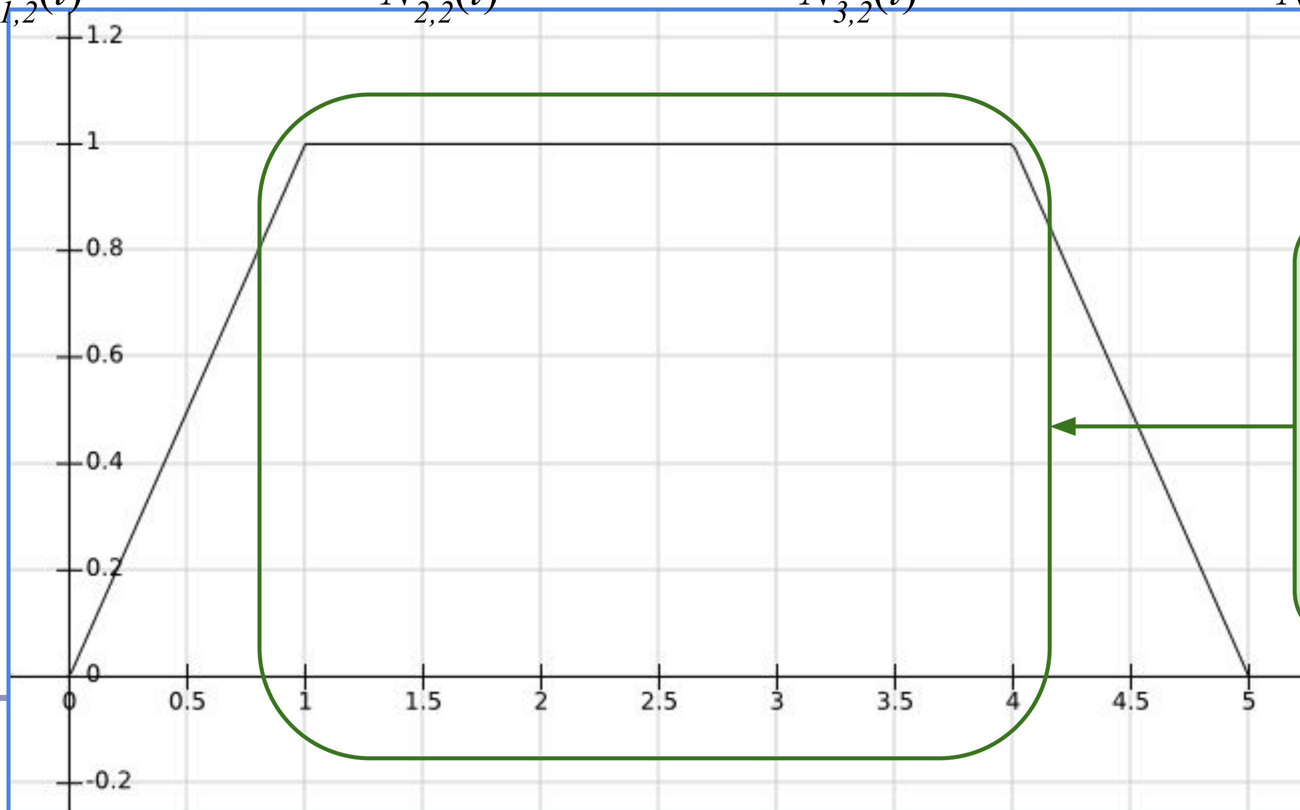
Knot vector = {0,1,2,3,4,5}, $k = 3 \rightarrow d = 2$ (degree = two)

Basis functions really sum to one (k=2)



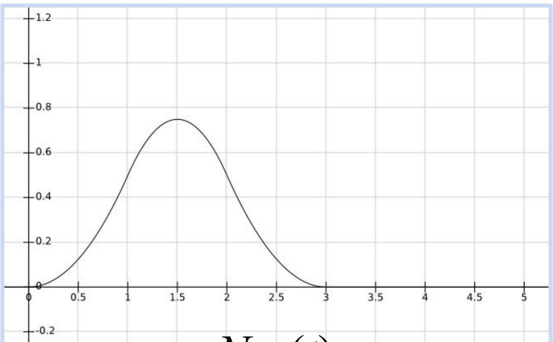
$N_{1,2}(t)$ + $N_{2,2}(t)$ + $N_{3,2}(t)$ + $N_{4,2}(t)$

=



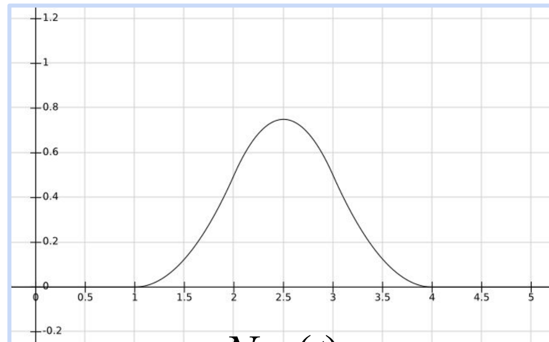
The sum of the four basis functions is fully defined (sums to one) between t_2 (t=1.0) and t_5 (t=4.0).

Basis functions really sum to one (k=3)



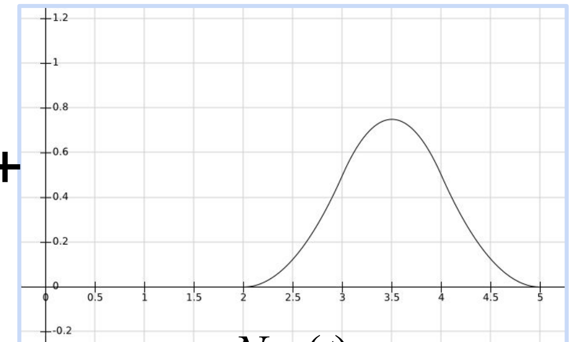
$N_{1,3}(t)$

+



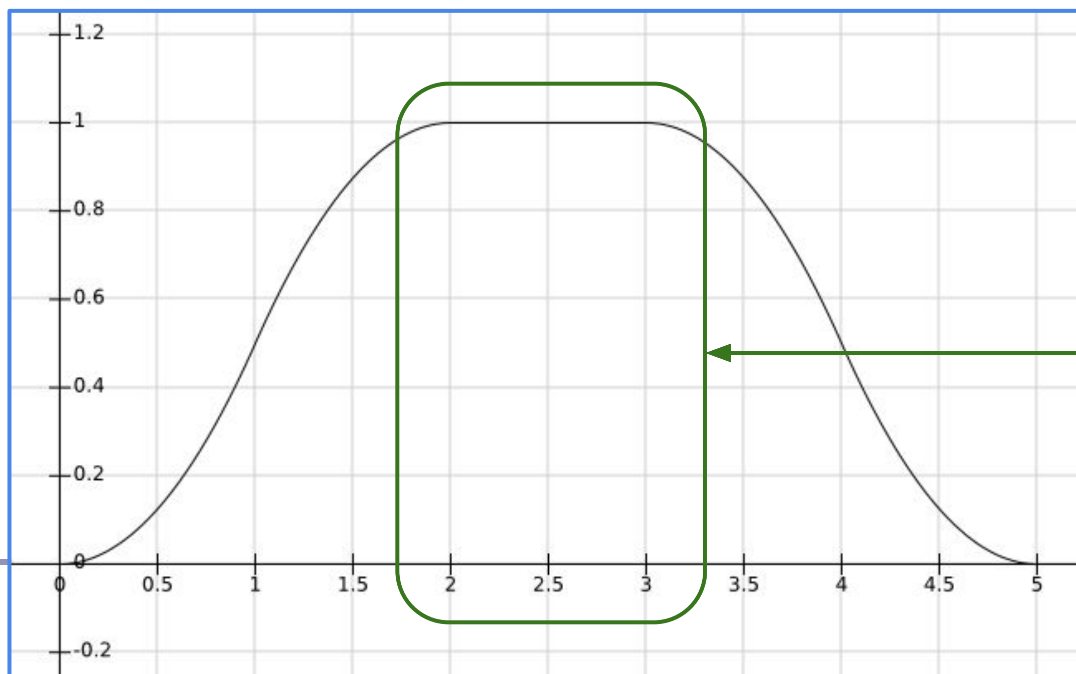
$N_{2,3}(t)$

+



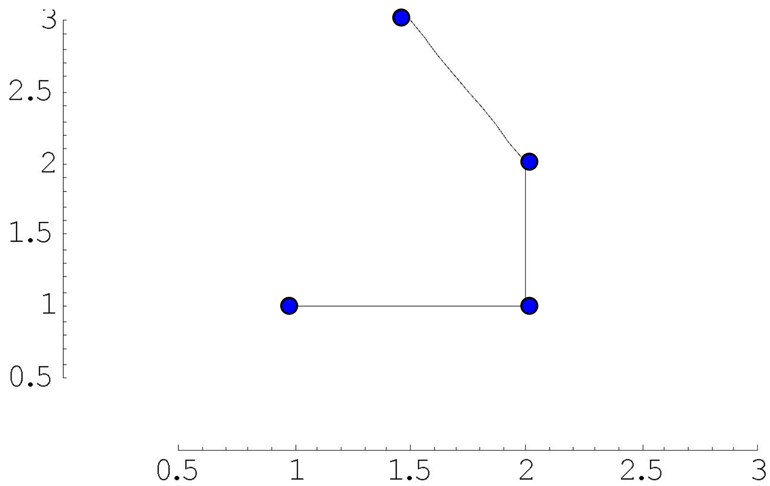
$N_{3,3}(t)$

=

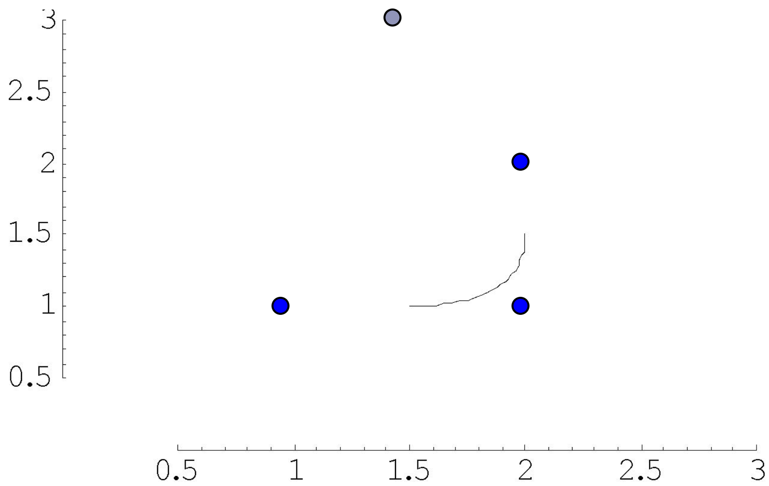


The sum of the three functions is fully defined (sums to one) between t_3 (t=2.0) and t_4 (t=3.0).

B-Splines



At k=2 the function is piecewise linear, depends on P_1,P_2,P_3,P_4, and is fully defined on $[t_2, t_5)$.



At k=3 the function is piecewise quadratic, depends on P_1,P_2,P_3, and is fully defined on $[t_3, t_4)$.

Each parameter-k basis function depends on $k+1$ knot values; $N_{i,k}$ depends on t_i through t_{i+k}, inclusive. So six knots → five discontinuous functions → four piecewise linear interpolations → three quadratics, interpolating three control points. n=3 control points, d=2 degree, k=3 parameter, n+k=6 knots.

Knot vector = {0,1,2,3,4,5}

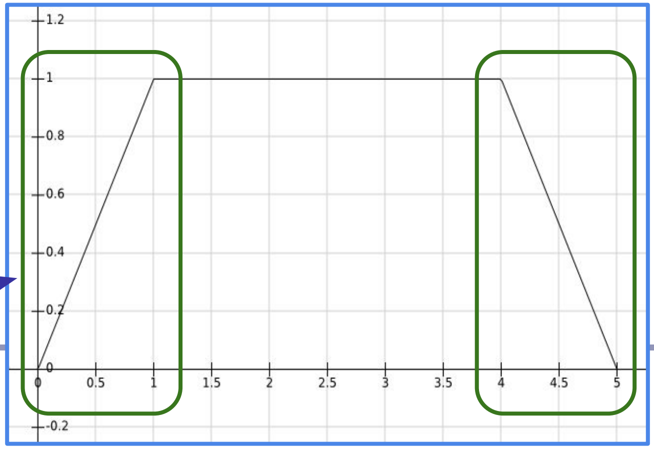
Non-Uniform B-Splines

- The knot vector {0,1,2,3,4,5} is *uniform*:

  $t_{i+1}-t_i = t_{i+2}-t_{i+1} \quad \forall\, t_i.$
- Varying the size of an interval changes the parametric-space distribution of the weights assigned to the control functions.
- Repeating a knot value reduces the continuity of the curve in the affected span by one degree.
- Repeating a knot *k* times will lead to a control function being influenced <u>only</u> by that knot value; the spline will pass through the corresponding control point with C0 continuity.

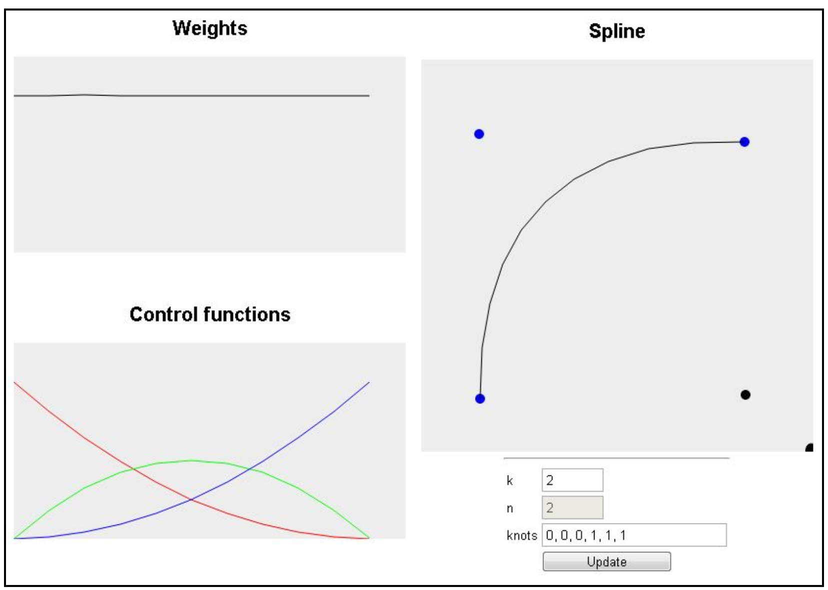
Open vs *Closed*

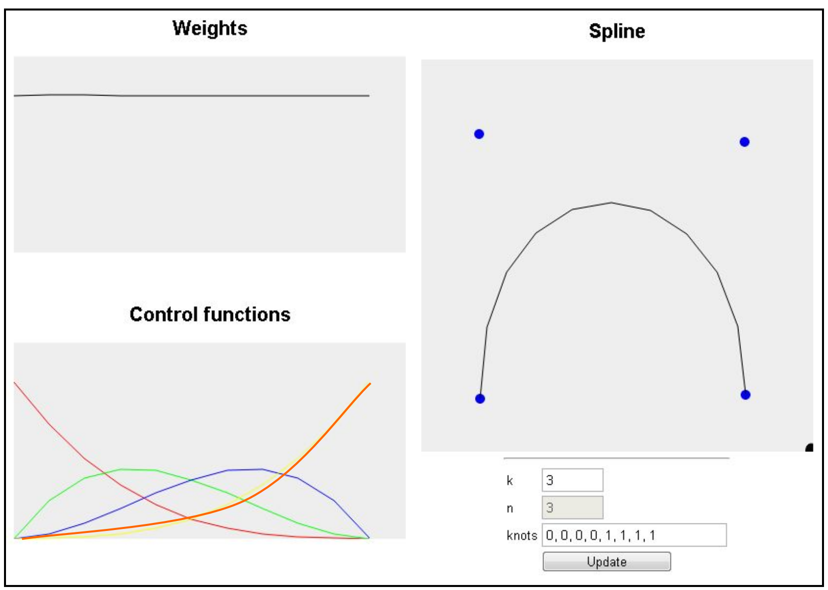
- A knot vector which repeats its first and last knot values k times is called *open* (or '*clamped*'), otherwise *closed*.
  - Repeating the knots k times is the only way to force the curve to pass through the first or last control point.
  - Without this, the functions $N_{1,k}$ and $N_{n,k}$ (which weight P_1 and P_n) would still be 'ramping up' and not yet equal to one at the first and last t.



Open vs Closed

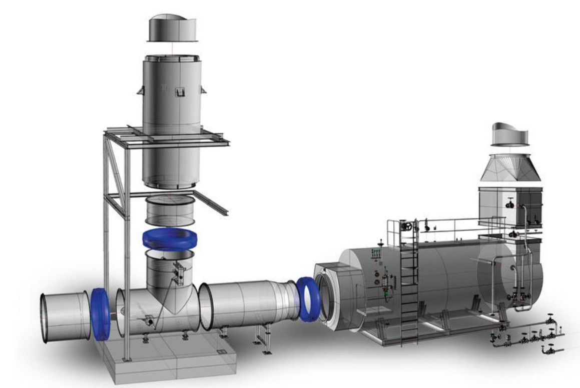
- Two open examples you may recognize:
  - k=3, n=3 control points, knots={0,0,0,1,1,1}
  - k=4, n=4 control points, knots={0,0,0,0,1,1,1,1}





NURBS curves

- *NURBS* (“*Non-Uniform Rational B-Splines*”) are a generalization of the Bezier curve concept:
  - NU: *Non-Uniform*. The knots in the knot vector are not required to be uniformly spaced.
  - R: *Rational*. The spline may be defined by rational polynomials (homogeneous coordinates.)
  - BS: ***B-Spline***. A generalization of Bezier splines with controllable degree.



Non-Uniform *Rational* B-Splines

- Repeating knot values is a clumsy way to control the curve's proximity to the control point.
  - We want to be able to slide the curve nearer or farther without losing continuity or introducing new control points.
  - The solution: *homogeneous coordinates*.
  - Associate a 'weight' with each control point: ω_i.

Non-Uniform Rational B-Splines

- Recall: $[x, y, z, \omega]_H \rightarrow [x / \omega, y / \omega, z / \omega]$
  - Or: $[x, y, z, 1] \rightarrow [x\omega, y\omega, z\omega, \omega]_H$
- The control point

  $P_i = (x_i, y_i, z_i)$

  becomes the homogeneous control point

  $P_{iH} = (x_i\omega_i, y_i\omega_i, z_i\omega_i)$
- A NURBS in homogeneous coordinates is:

$$P_H(t) = \sum_{i=1}^{n} N_{i,k}(t) P_{iH}, \; t_{min} \leq t < t_{max}$$

Non-Uniform Rational B-Splines

- To convert from homogeneous coords to normal coordinates:

$$
\begin{aligned}
x_H(t) &= \textstyle\sum_{i=1}^{n}(x_i\omega_i)(N_{i,k}(t)) \\
y_H(t) &= \textstyle\sum_{i=1}^{n}(y_i\omega_i)(N_{i,k}(t)) \\
z_H(t) &= \textstyle\sum_{i=1}^{n}(z_i\omega_i)(N_{i,k}(t)) \\
\omega(t) &= \textstyle\sum_{i=1}^{n}(\omega_i)(N_{i,k}(t))
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= x_H(t)/\omega(t) \\
y(t) &= y_H(t)/\omega(t) \\
z(t) &= z_H(t)/\omega(t)
\end{aligned}
$$

Non-Uniform Rational B-Splines

- A piecewise rational curve is thus defined by:

  $$P(t) = \sum_{i=1}^{n} R_{i,k}(t) P_i, \; t_{min} t < t_{max}$$

  with supporting *rational basis functions*:

  $$R_{i,k}(t) = \frac{\omega_i N_{i,k}(t)}{\sum_{j=1}^{n} \omega_j N_{j,k}(t)}$$

  This is essentially an average re-weighted by the ω's.
- Such a curve can be made to pass arbitrarily far or near to a control point by changing the corresponding weight.

References

- Les Piegl and Wayne Tiller, *The NURBS Book*, Springer (1997)
- Alan Watt, *3D Computer Graphics*, Addison Wesley (2000)
- G. Farin, J. Hoschek, M.-S. Kim, *Handbook of Computer Aided Geometric Design*, North-Holland (2002)